Generating and Searching Families of FFT Algorithms

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Overview

- Proof of the lowest operation count for classes of discrete Fourier transforms
  - Require fixed flowgraph structure of common FFTs
  - Require all complex multiplication by n^{th} roots of unity
- Found new FFTs with lower FLOP count than split-radix
  - Undiscovered in past 40 years despite intense study
- Technique is exhaustive and supports various search objectives
- Full paper to appear in JSAT
  - Preprint at www.arXiv.org
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1. The Fast Fourier Transform
2. Generating a Family of FFT Algorithms
3. Searching a Family of FFT Algorithms
4. Results and Conclusions

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Fourier Transform

- Fourier Transform is an Integral

$$X(f) = \int_{-\infty}^{\infty} a(t) e^{-i2\pi ft} dt, \quad f \in (-\infty, \infty)$$

- But a discrete sum is used to compute the Fourier Transform

$$\begin{aligned} X(k) &= \sum_{j=0}^{n-1} a_j e^{-\frac{i2\pi}{n} jk} \\ &= \sum_{j=0}^{n-1} a_j \omega_n^{jk \pmod n}, \quad k = 0, 1, 2, \ldots, n-1 \end{aligned}$$

Multiplication Example

$$\begin{aligned} \omega_{16}^{13} \omega_{16}^{6} &= \omega_{16}^{(13+6 \pmod{16})} \\ &= \omega_{16}^{3} \end{aligned}$$

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Expanded 8-Point Discrete Fourier Transform

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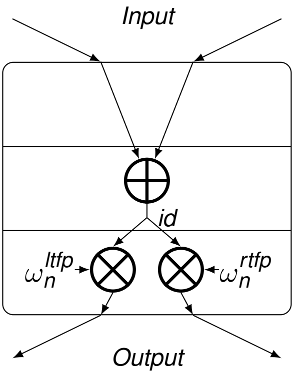
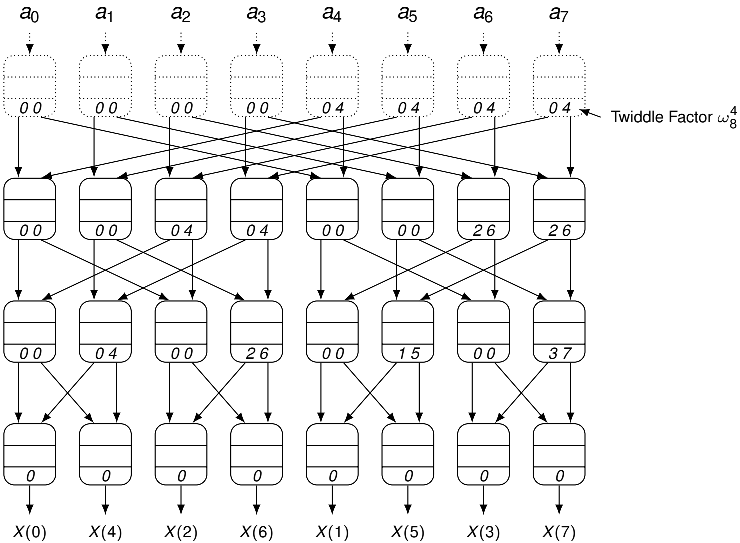
Graph Vertex Internals



Classic Fast Fourier Transform



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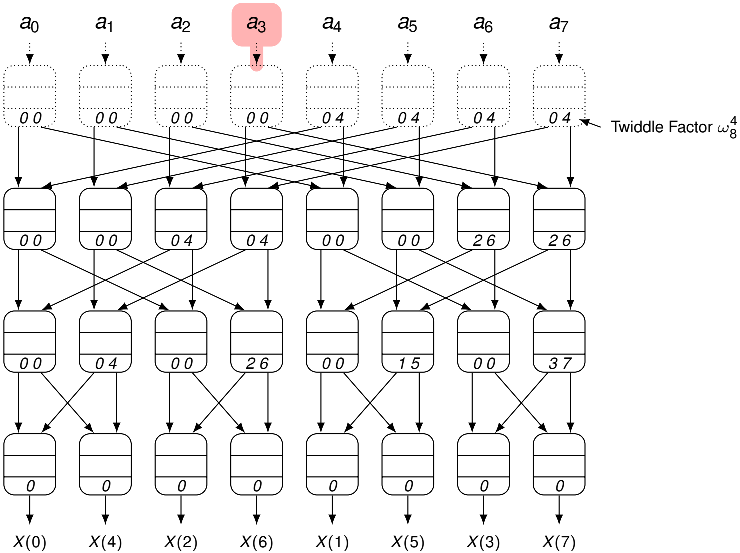

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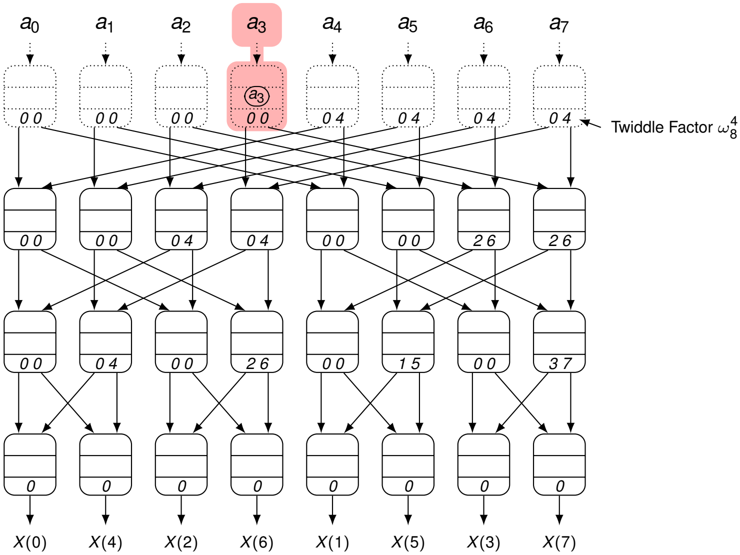

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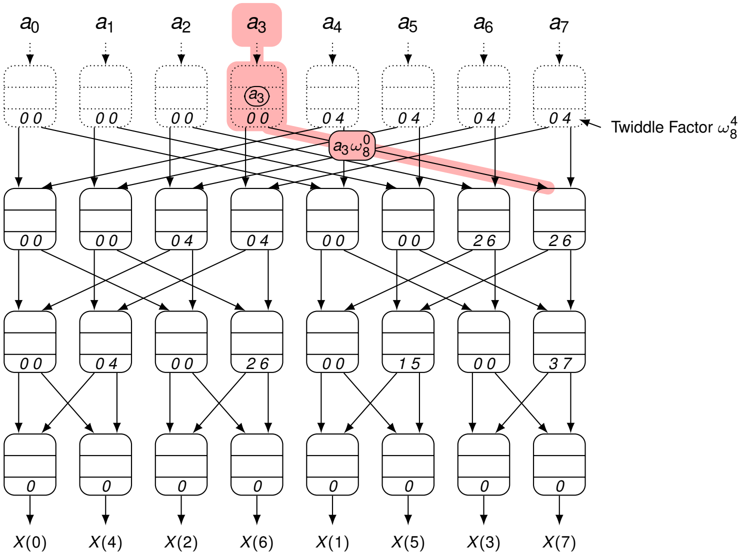

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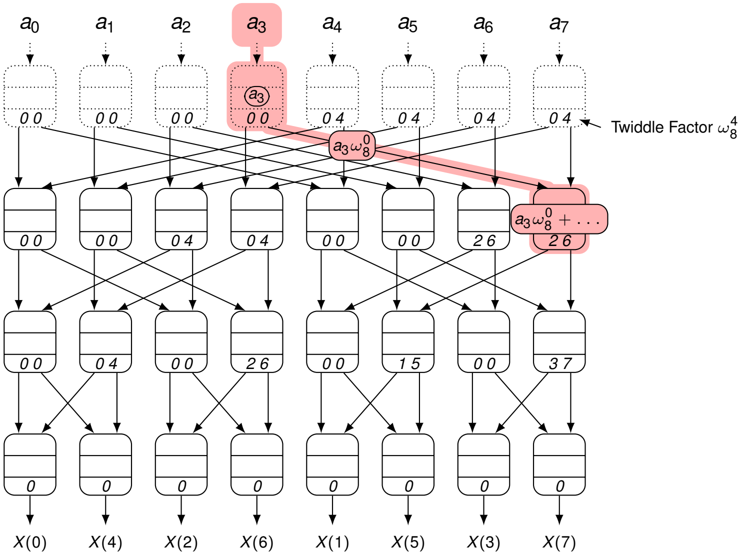

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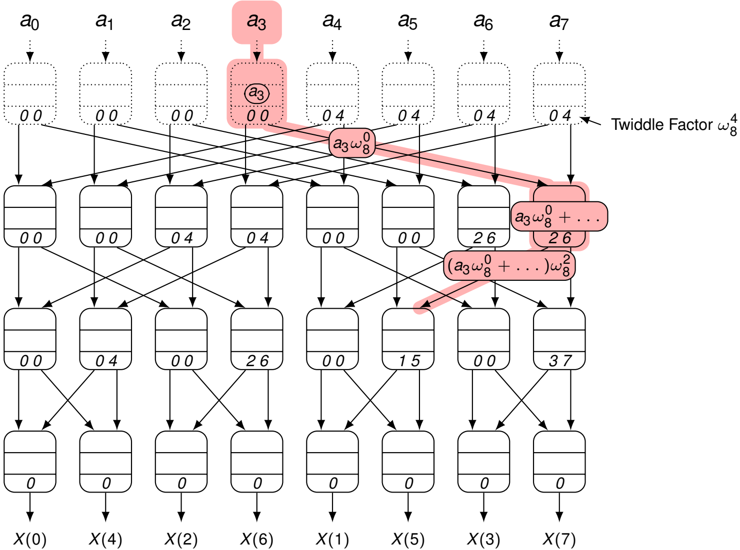

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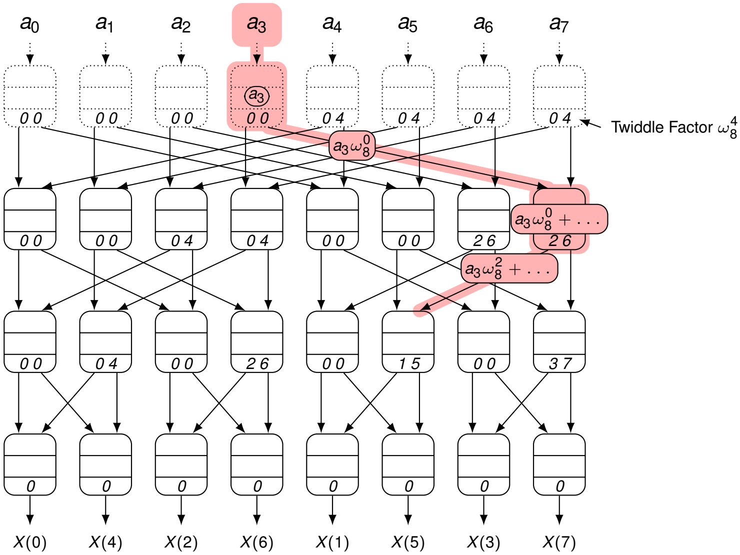

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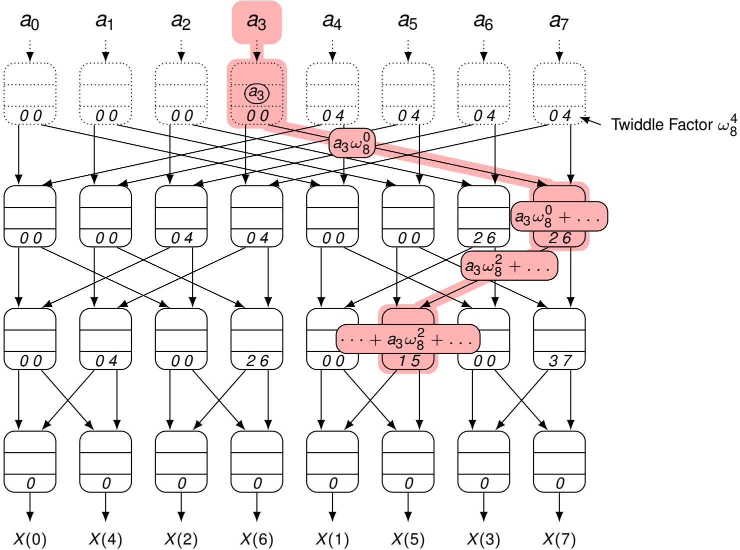

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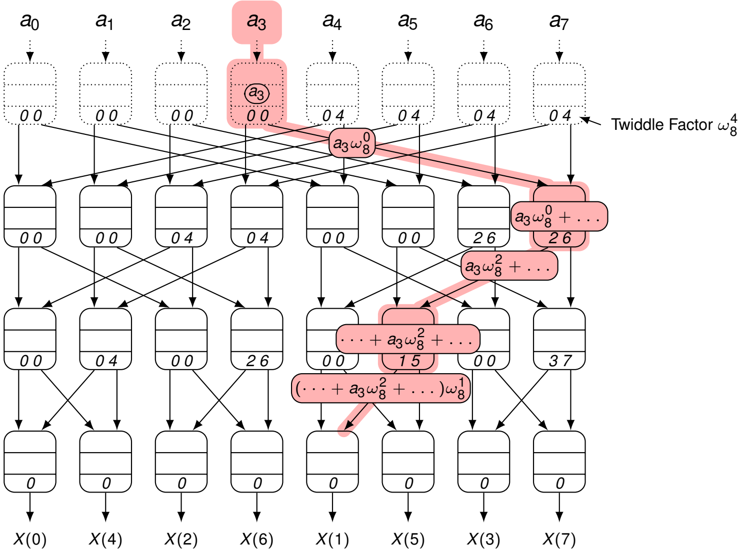

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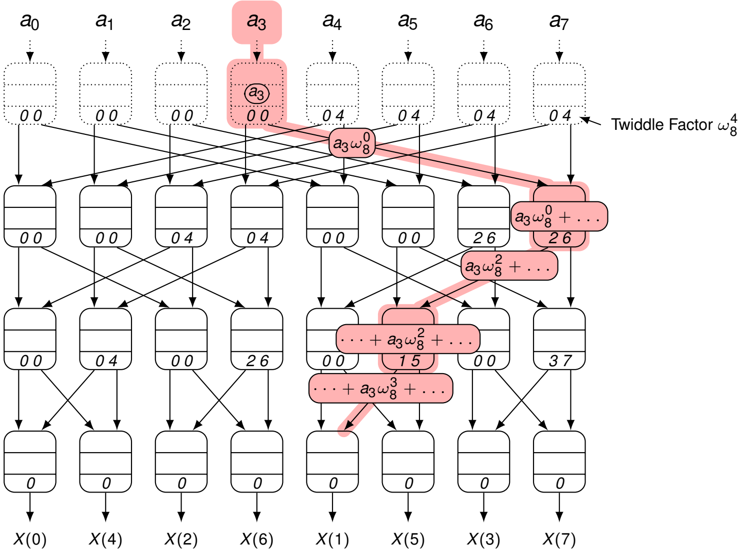

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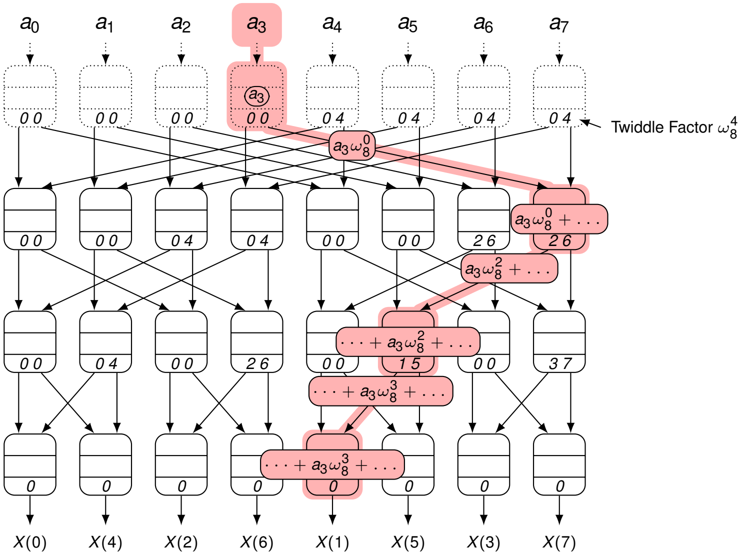

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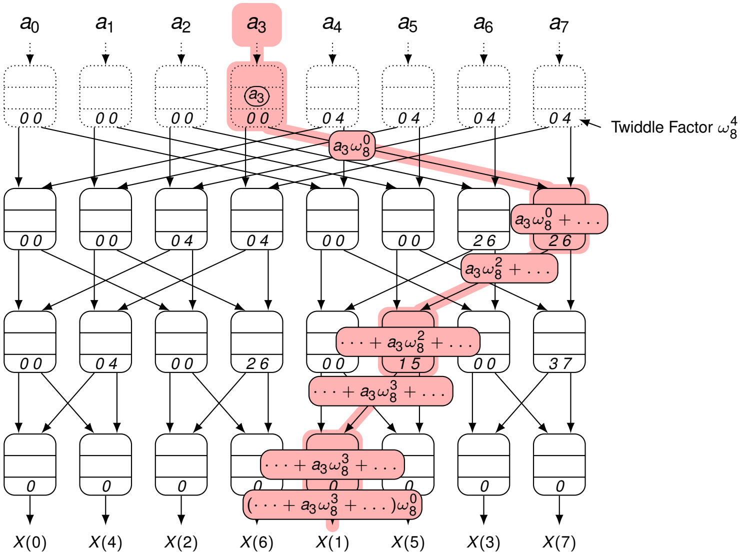

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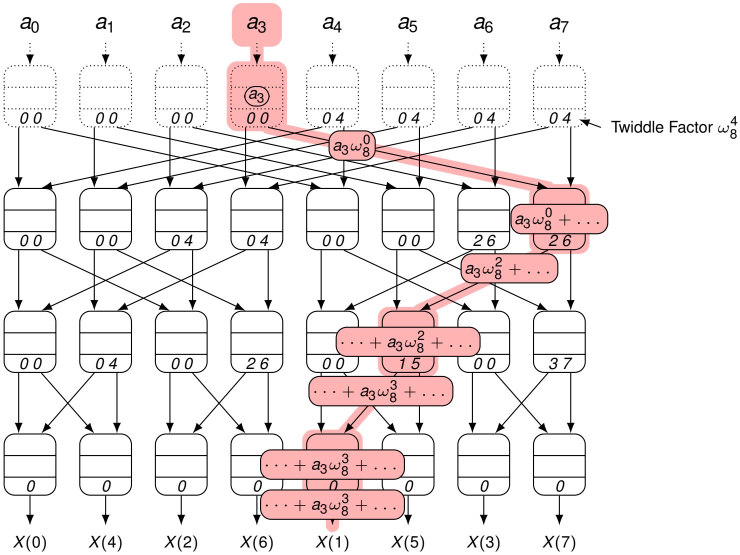

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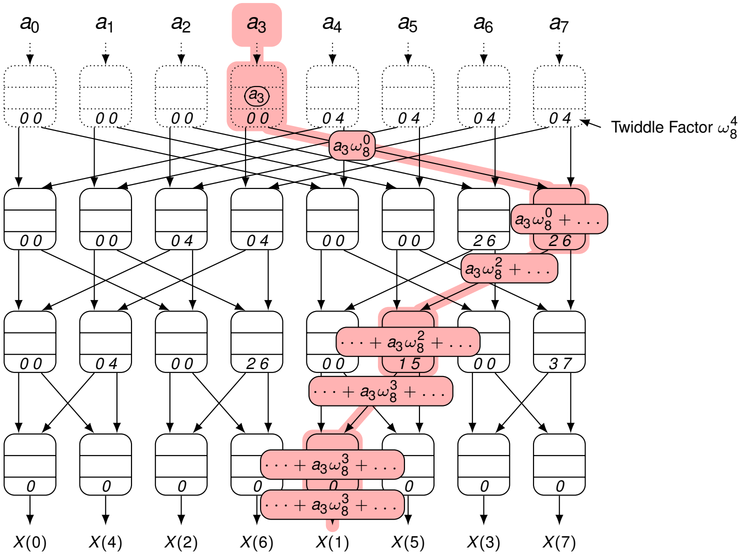

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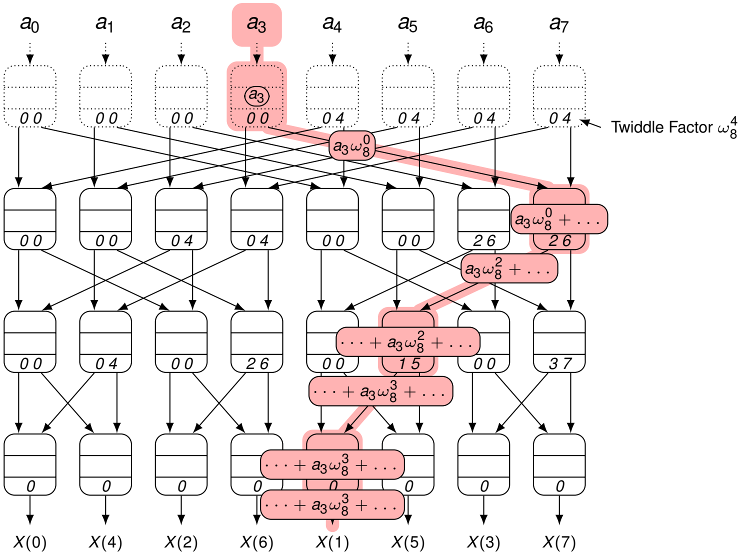
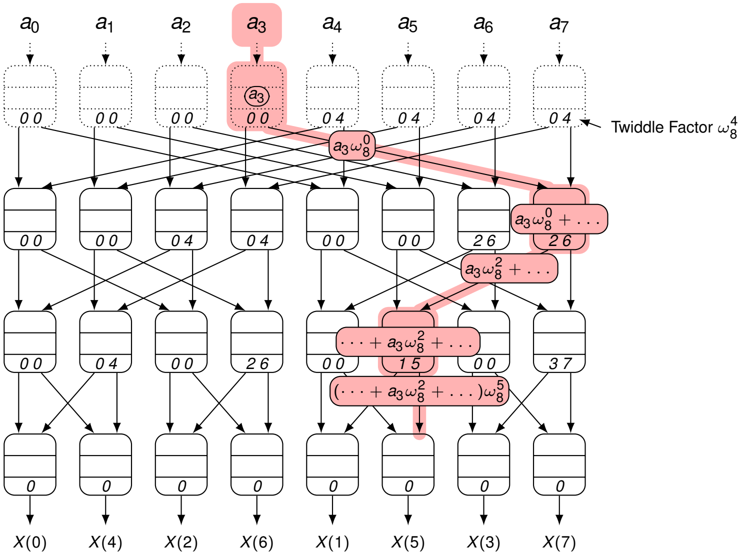


$X(1) = \dots + a_3\omega_8^3 + \dots$

Classic Fast Fourier Transform



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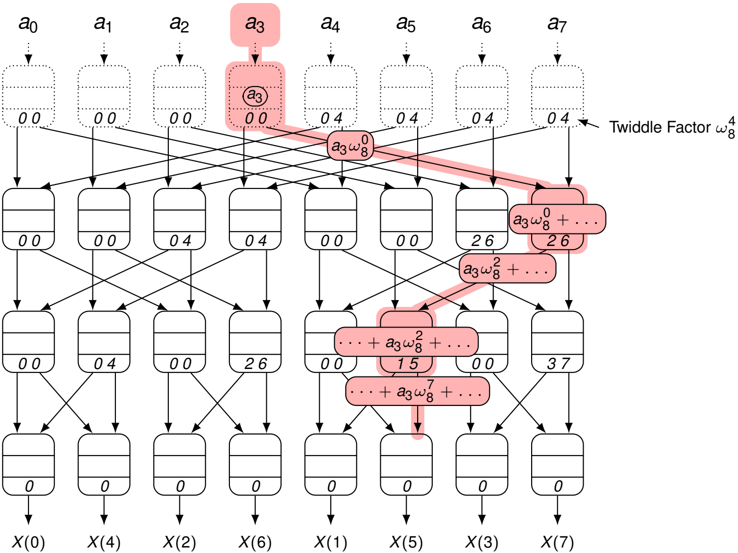

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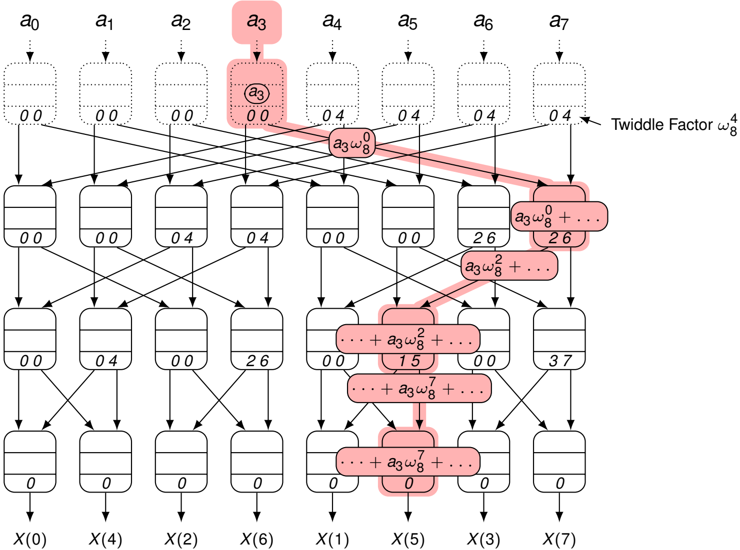

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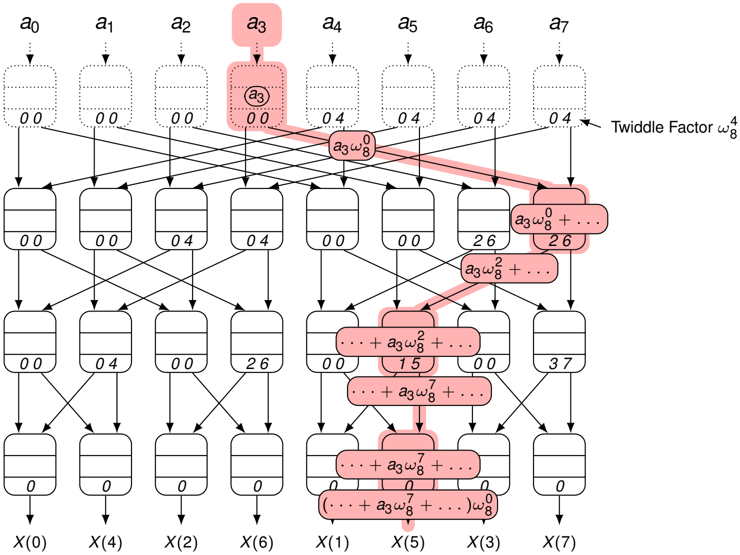

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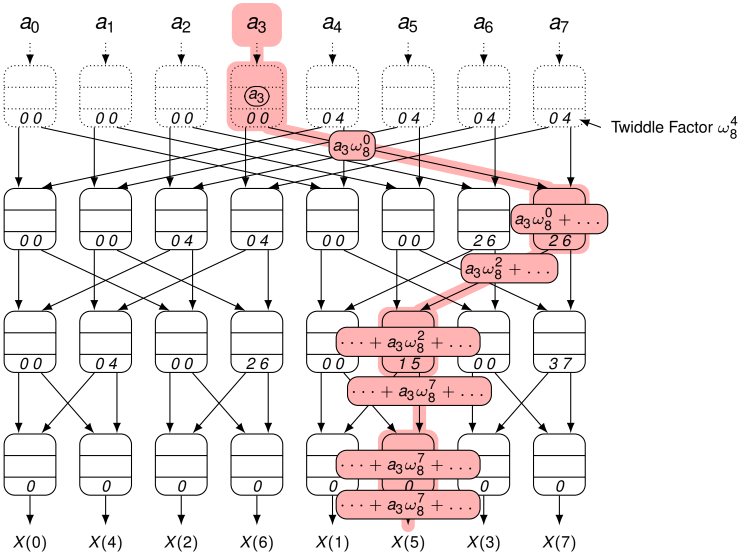

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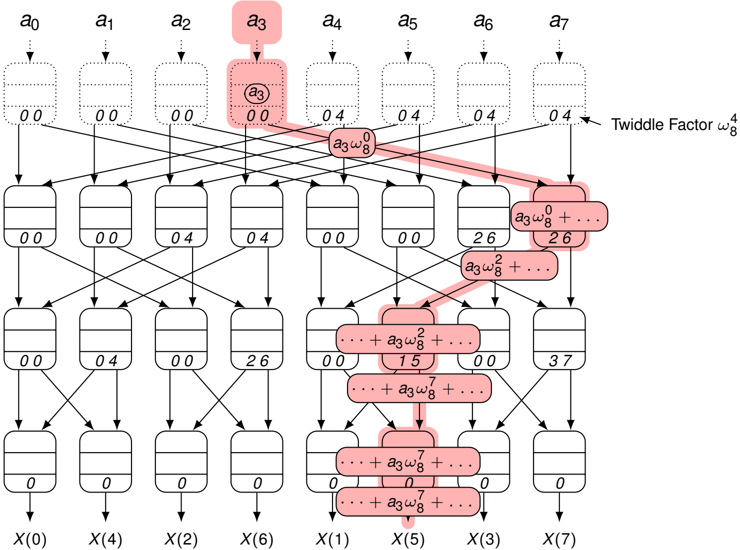
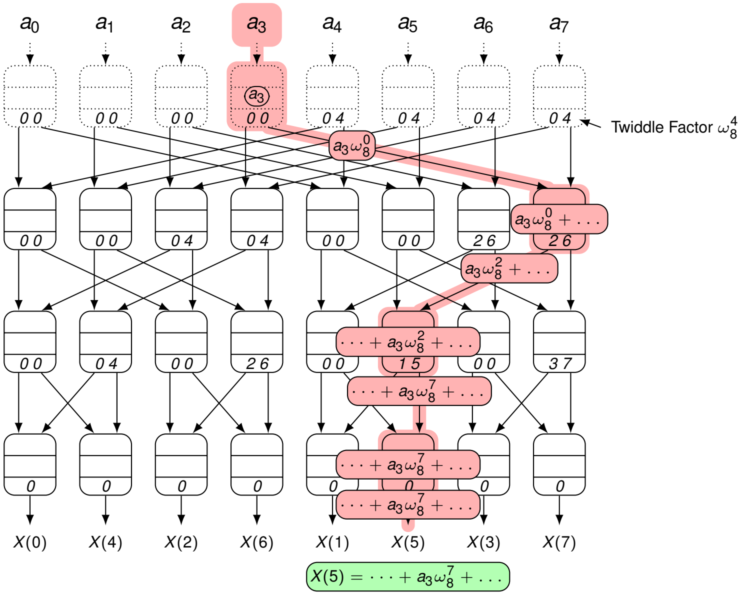


$$X(5) = a_0\omega_8^0 + a_1\omega_8^5 + a_2\omega_8^2 + a_3\omega_8^7 + a_4\omega_8^4 + a_5\omega_8^1 + a_6\omega_8^6 + a_7\omega_8^3$$

Classic Fast Fourier Transform



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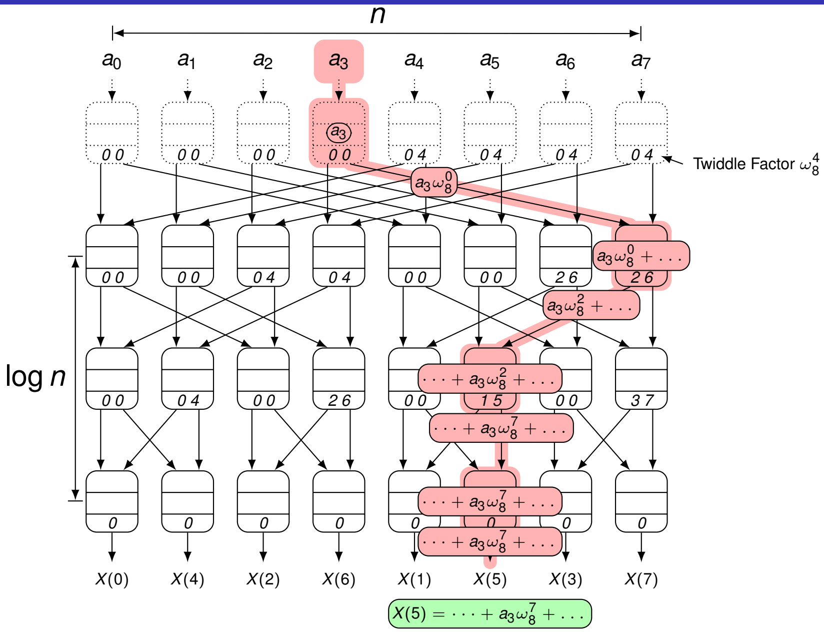

The Costs of Calculating Complex Multiplication

- A computer represents complex numbers as $a + bi$

$$
\begin{aligned}
z &= (a_1 + b_1 i)(a_2 + b_2 i) \\
&= a_1 a_2 + a_1 b_2 i + b_1 a_2 + b_1 b_2 i^2 \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \\
\mathfrak{Re}(z) &= (a_1 a_2 - b_1 b_2) \\
\mathfrak{Im}(z) &= (a_1 b_2 + b_1 a_2) i
\end{aligned}
$$

- $\mathfrak{Re}(z)$ requires 2 real multiplications and 1 real addition
- $\mathfrak{Im}(z)$ also requires 2 real multiplications and 1 real addition
- 6 floating point operations (FLOPS) required for a complete complex multiplication

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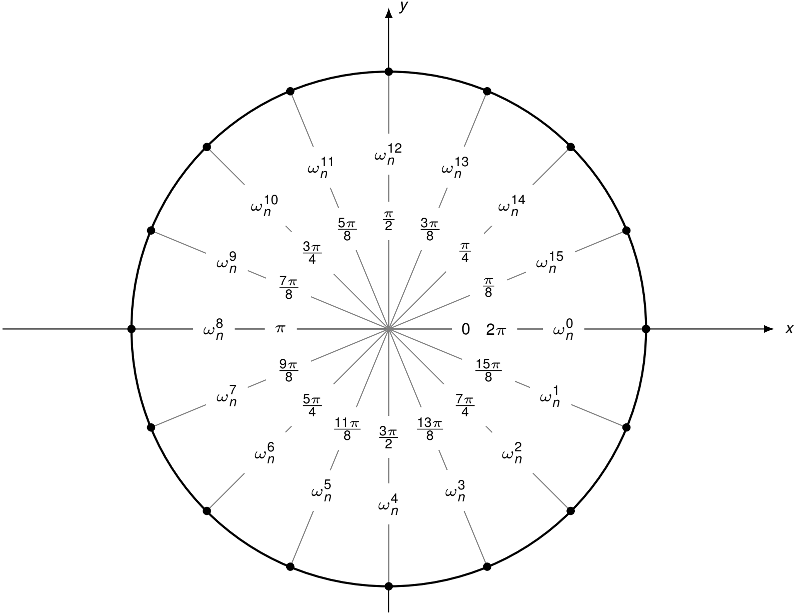
Not All Complex Multiplications Cost the Same



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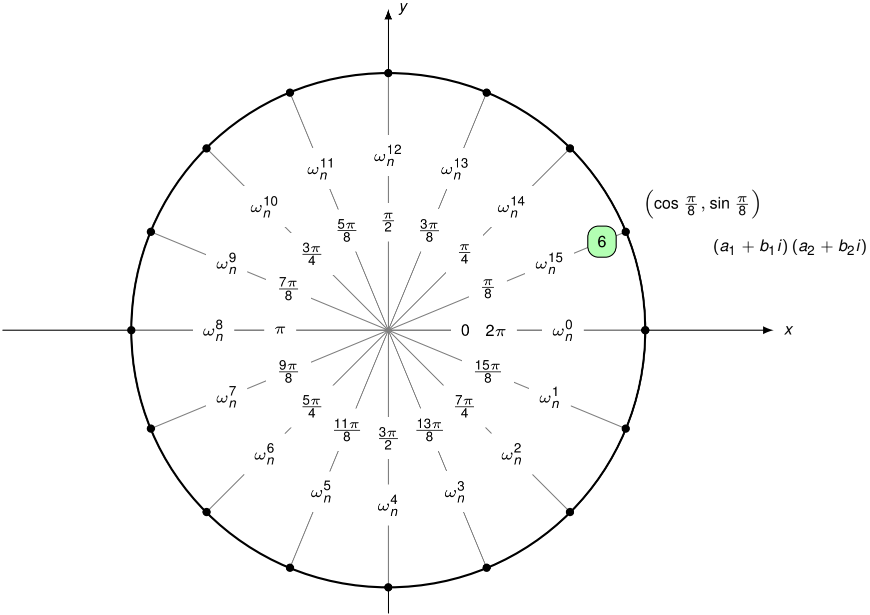

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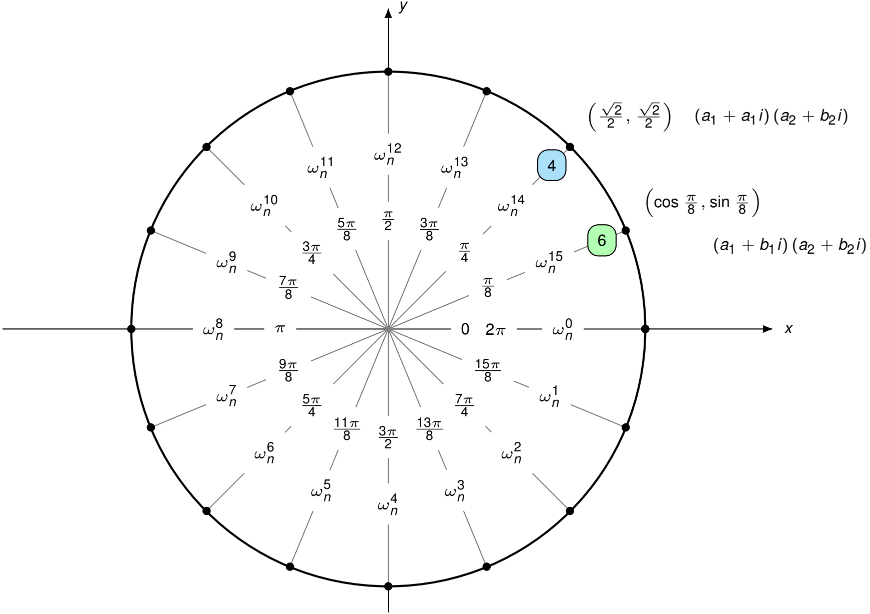

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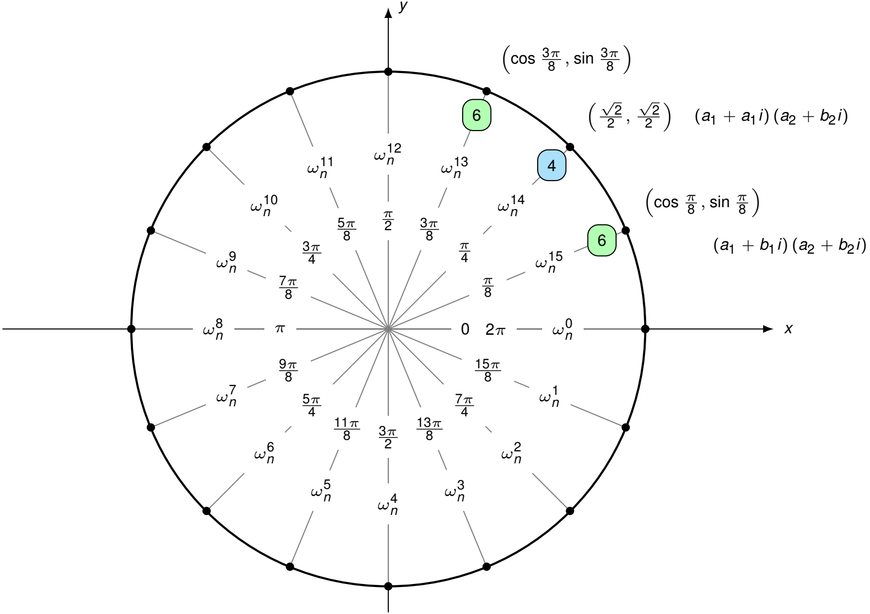

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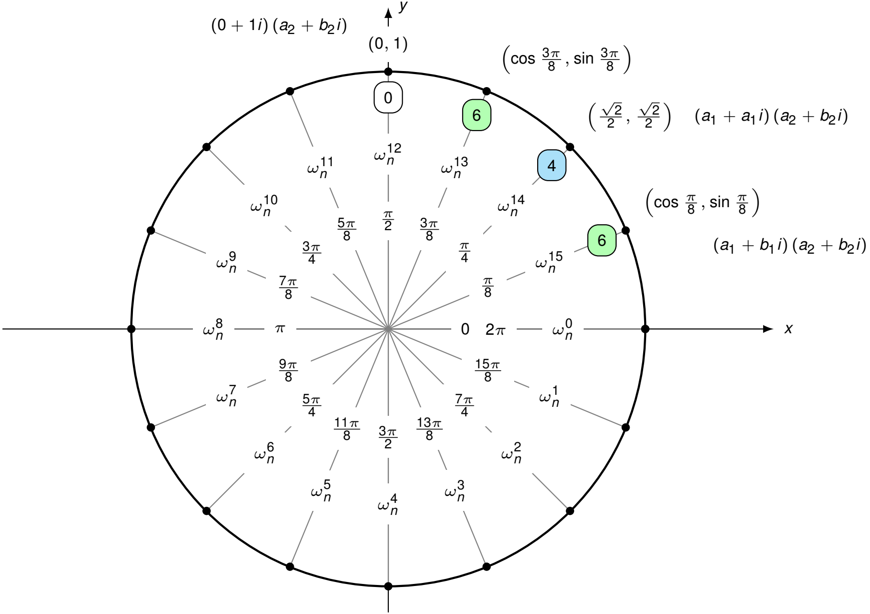

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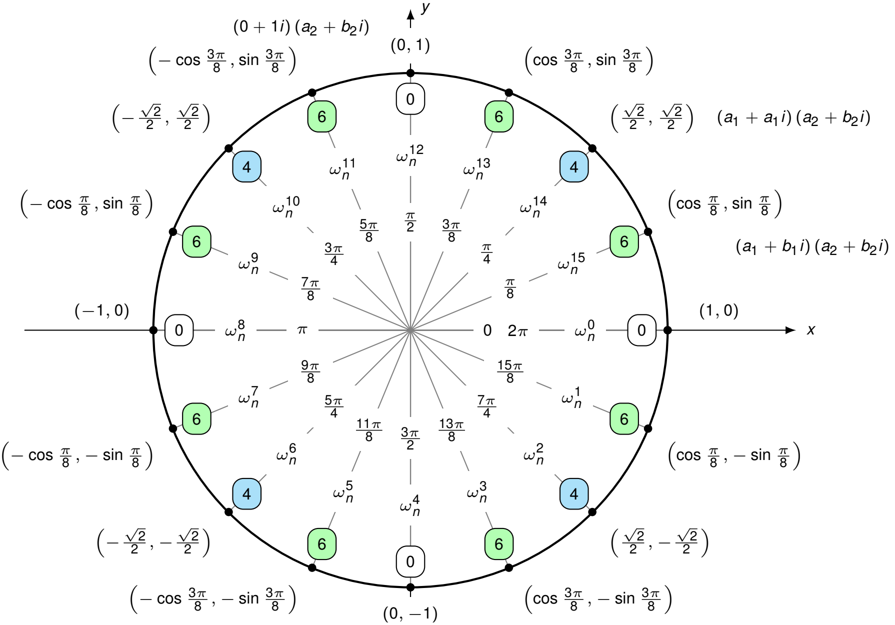
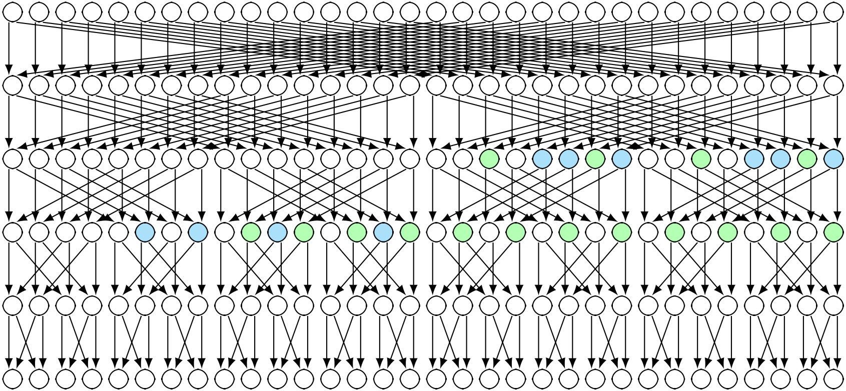
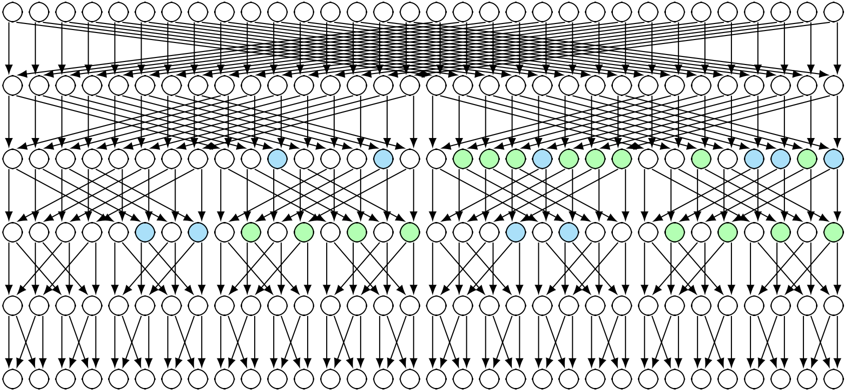


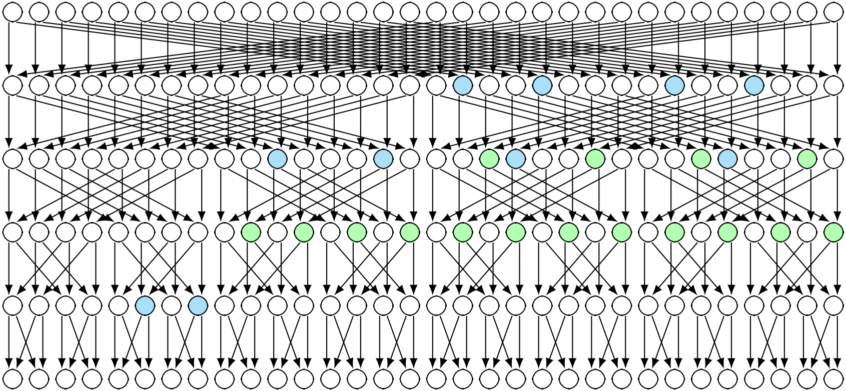
32-Point FFT requiring 456 FLOPs



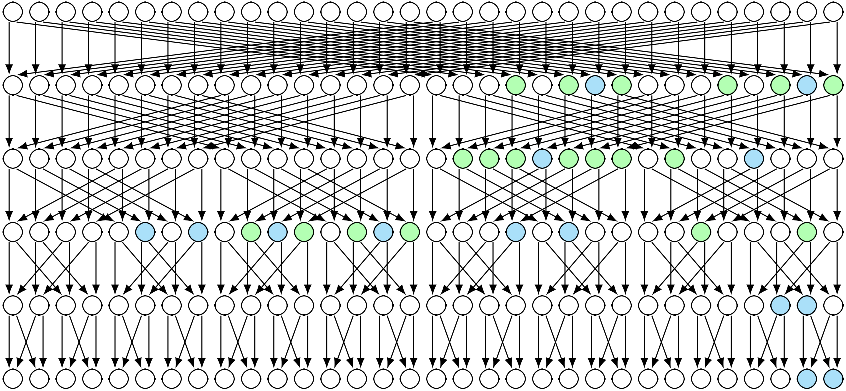
Another 32-Point FFT requiring 456 FLOPs



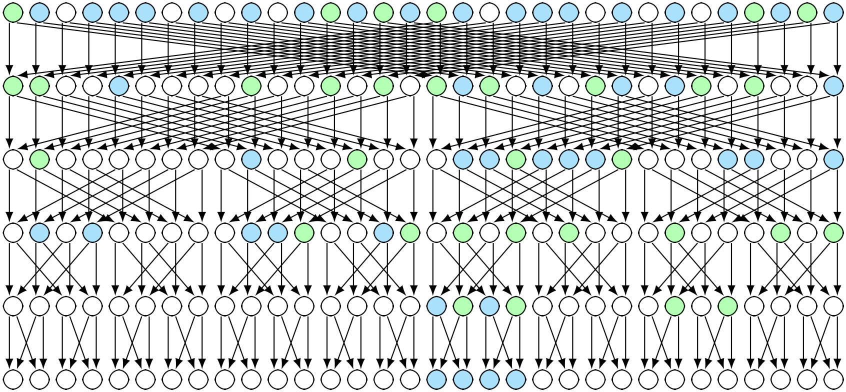
Yet Another 32-Point FFT requiring 456 FLOPs



32-Point FFT requiring 490 FLOPs

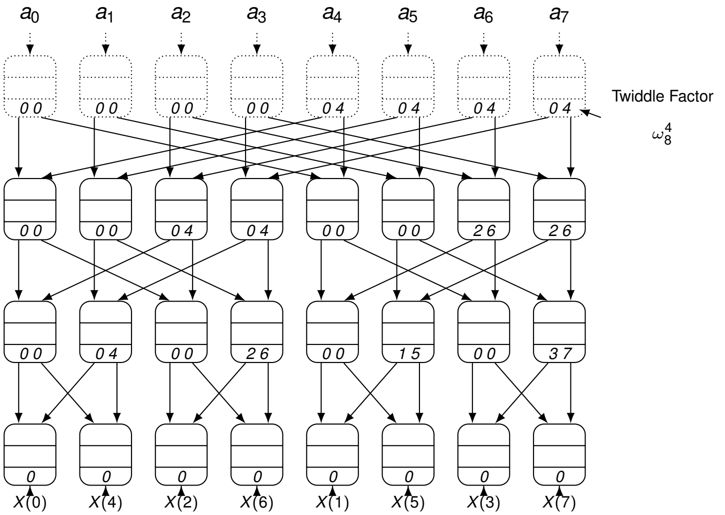


32-Point FFT requiring 688 FLOPs



Outline

Weight Stride Invariant



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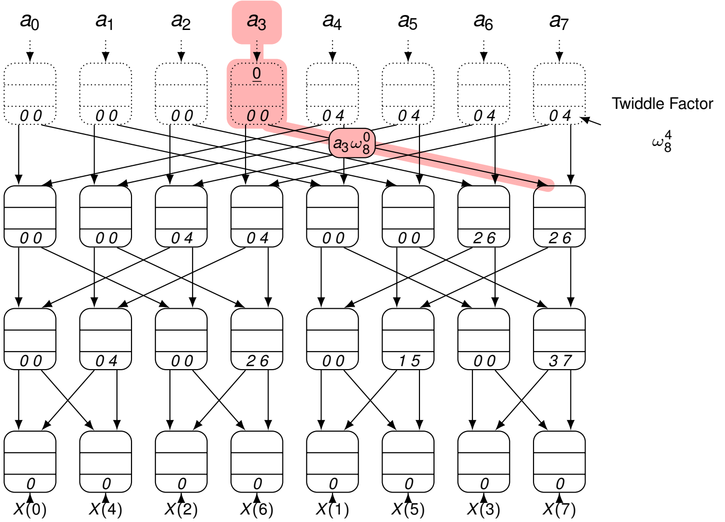

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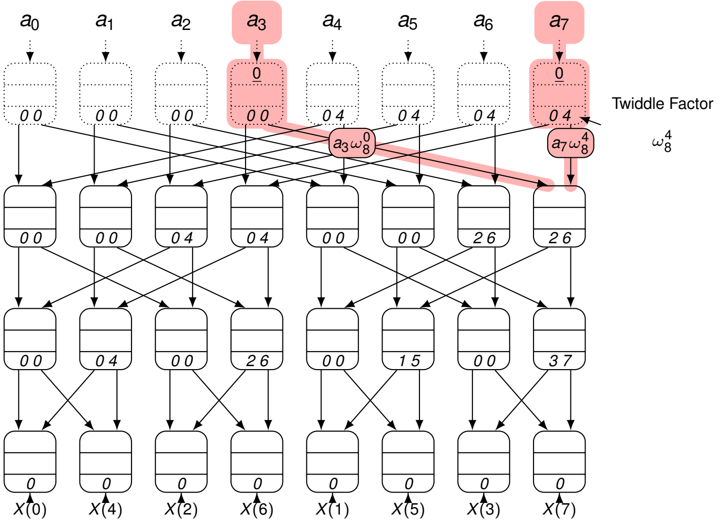

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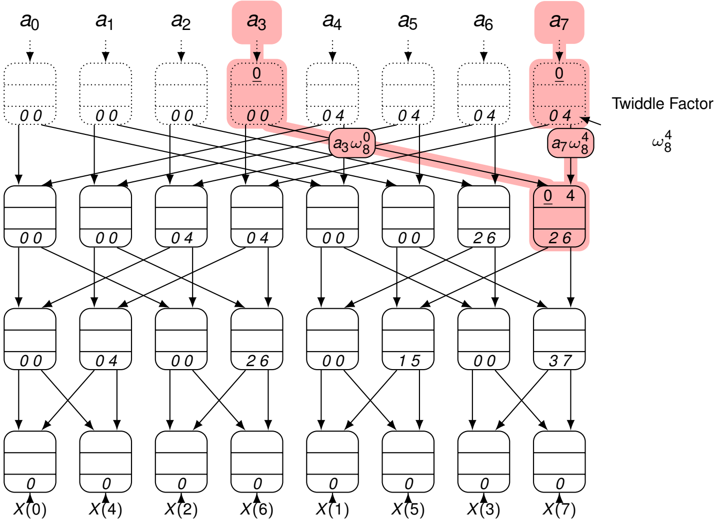

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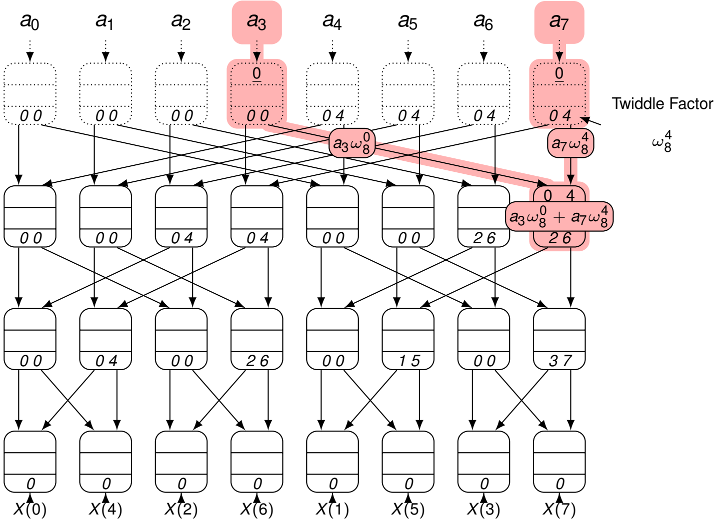

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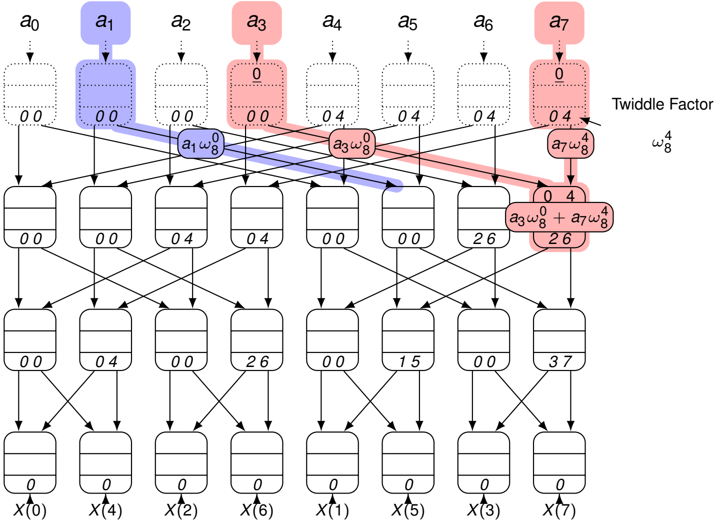

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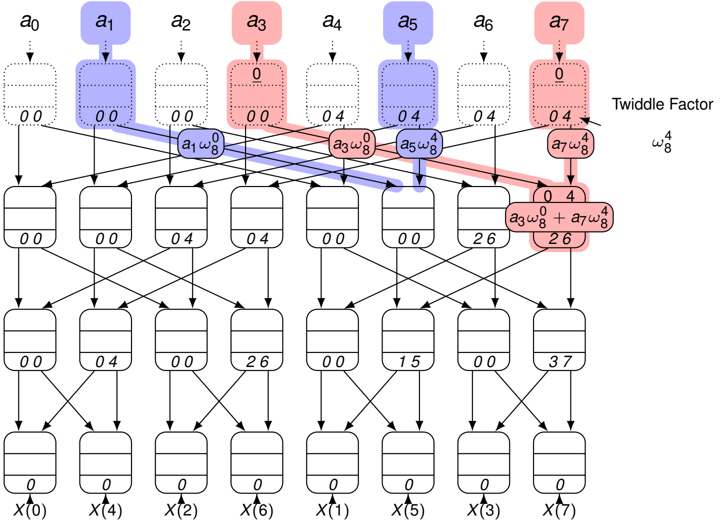

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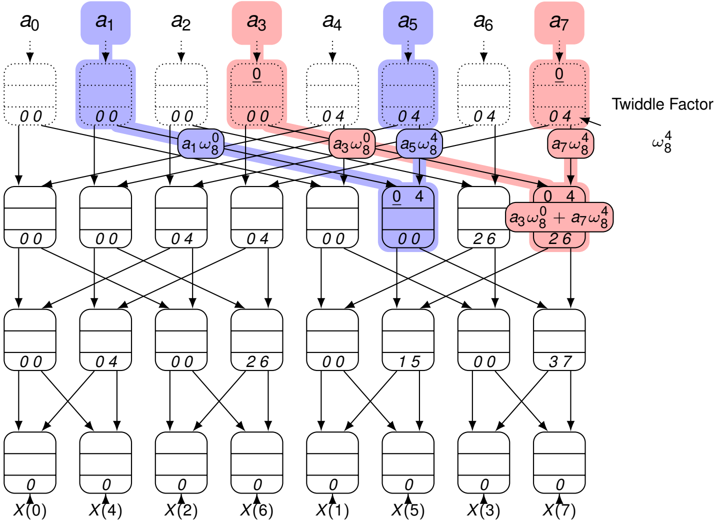

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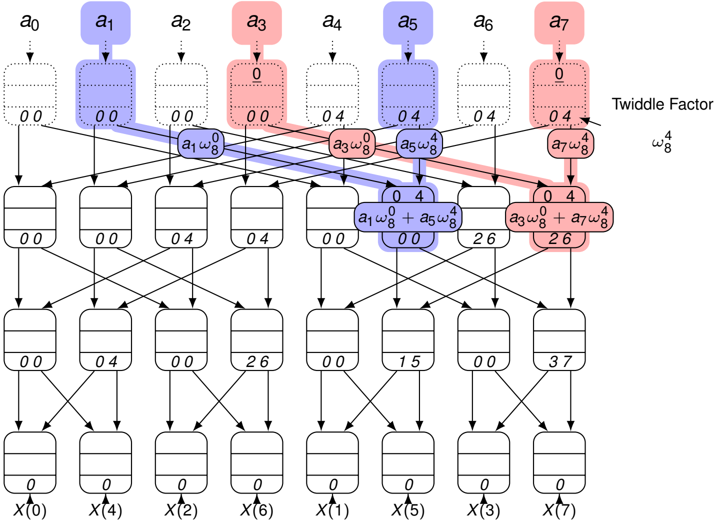

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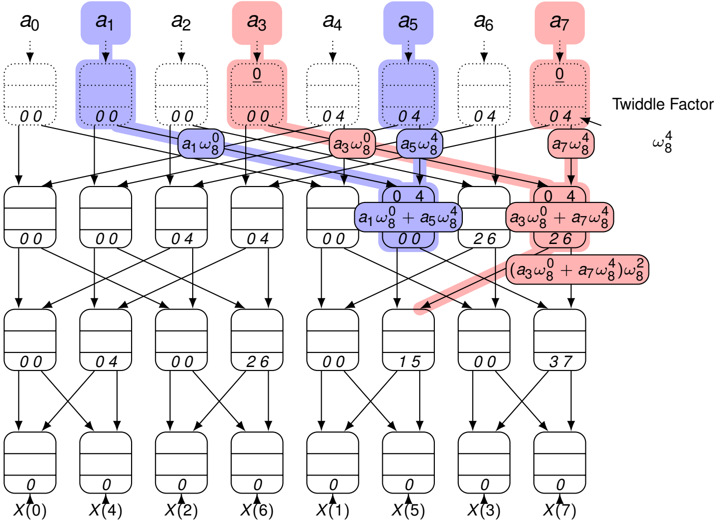

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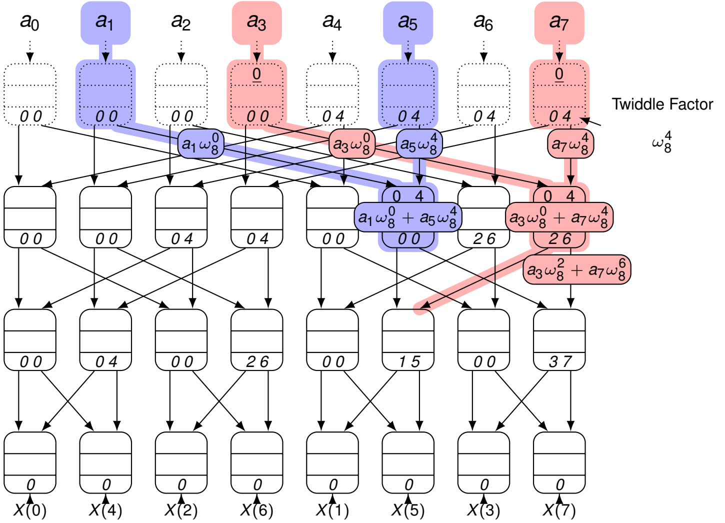

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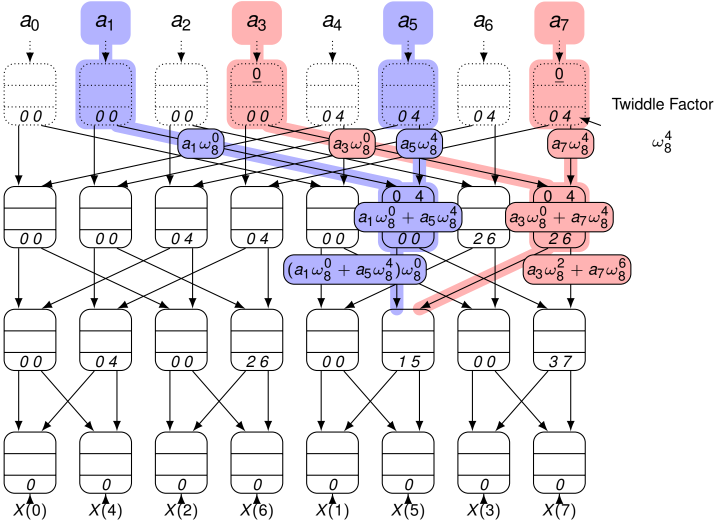

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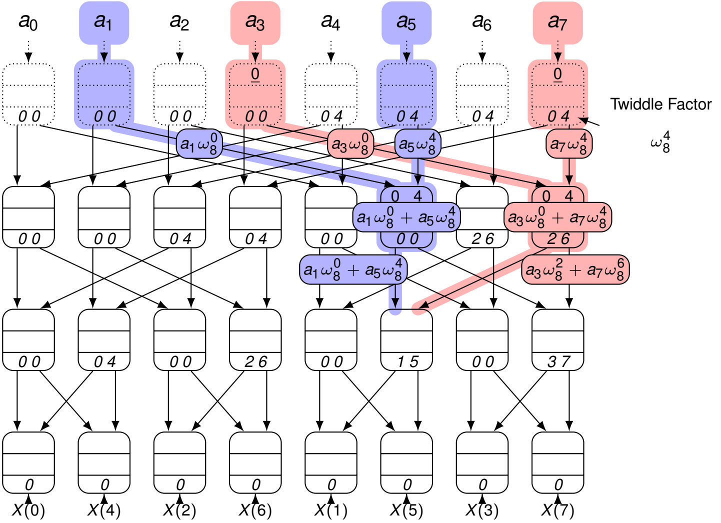

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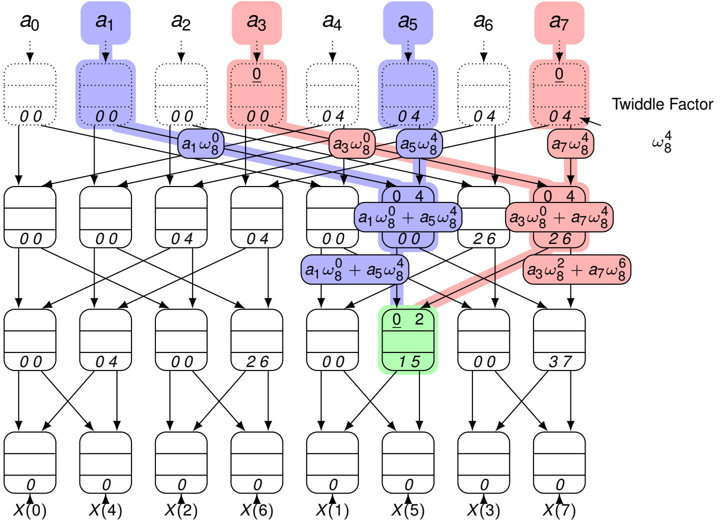

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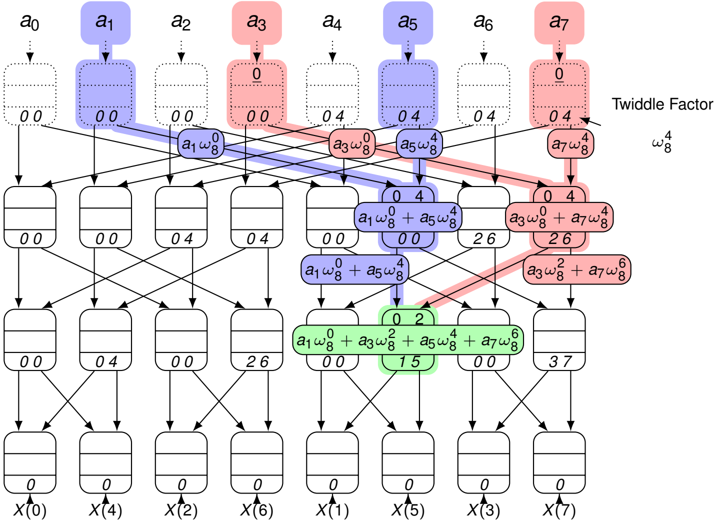

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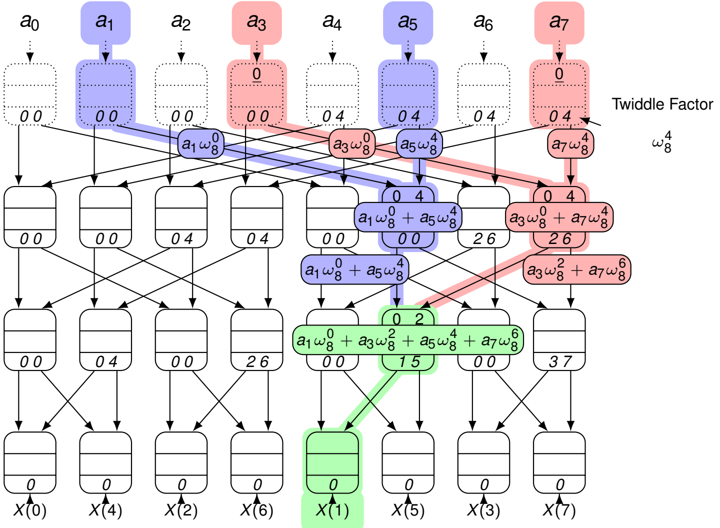

$$X(1) = a_0\omega_8^0 + a_1\omega_8^1 + a_2\omega_8^2 + a_3\omega_8^3 + a_4\omega_8^4 + a_5\omega_8^5 + a_6\omega_8^6 + a_7\omega_8^7$$

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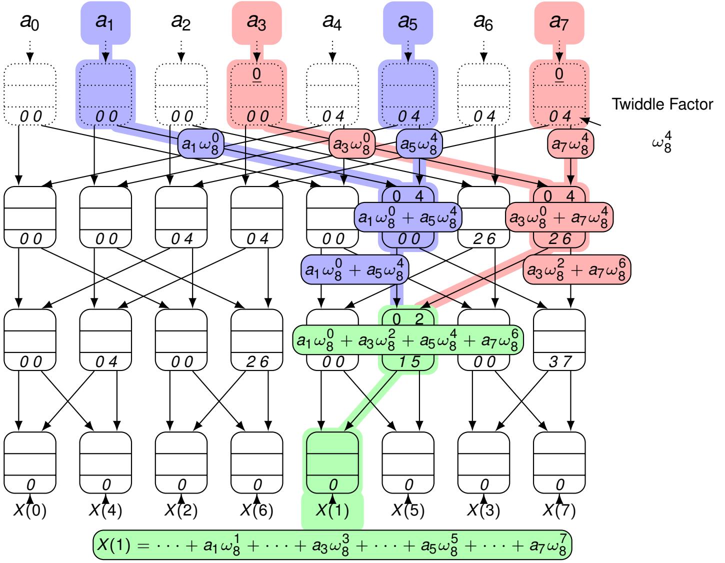

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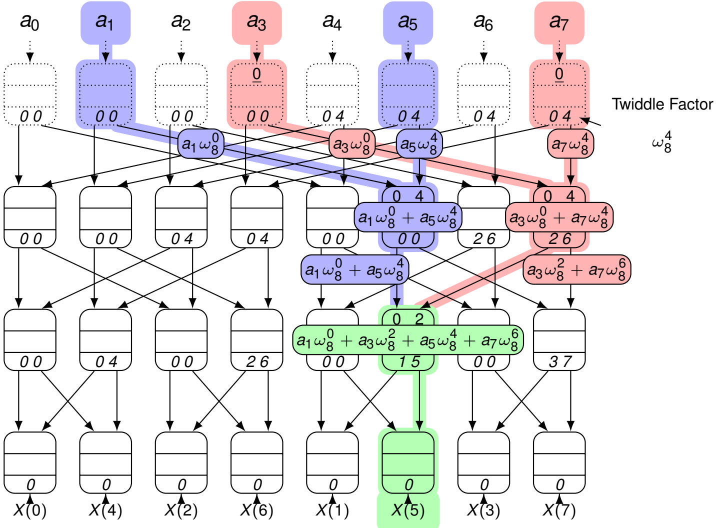

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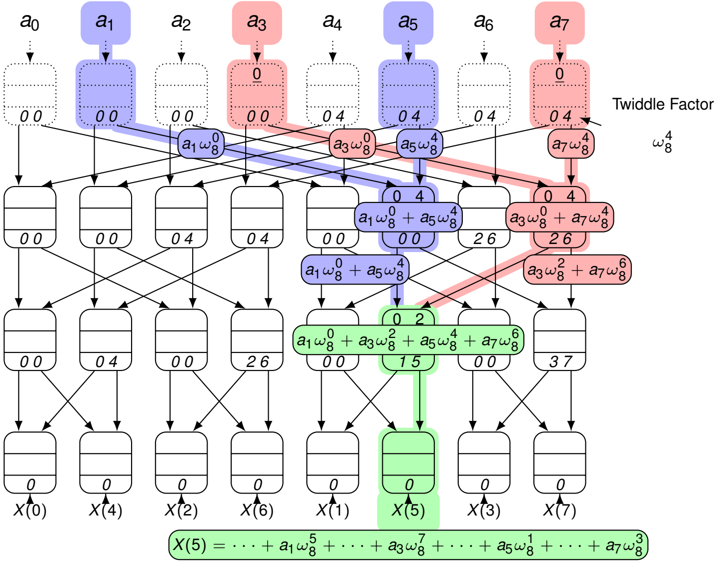

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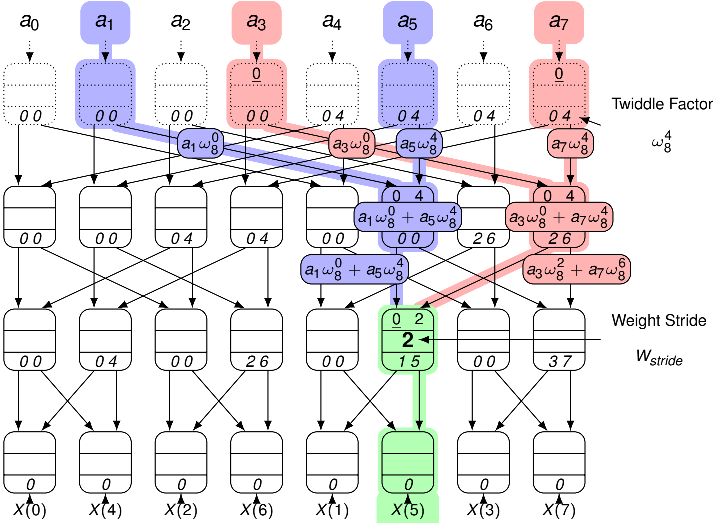
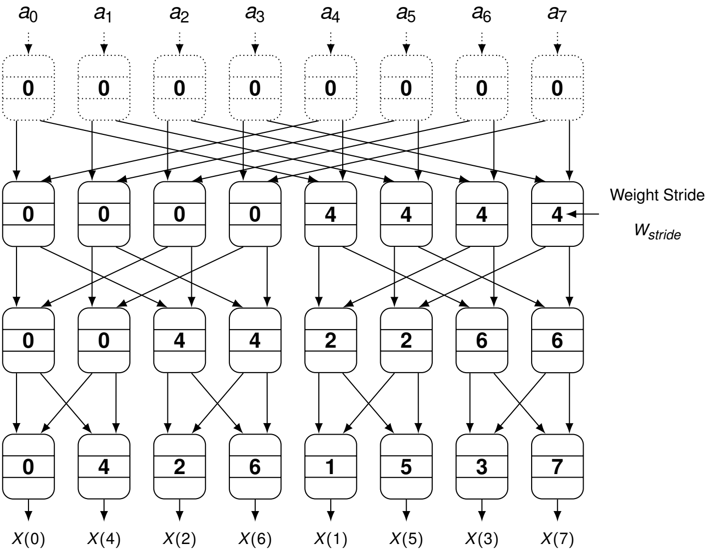


$X(5) = \cdots + a_1\omega_8^5 + \cdots + a_3\omega_8^7 + \cdots + a_5\omega_8^1 + \cdots + a_7\omega_8^3$

A Random Family Member



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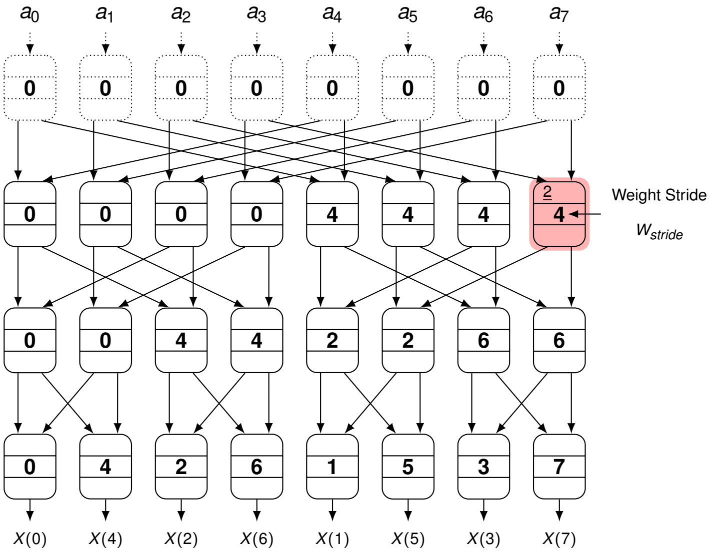

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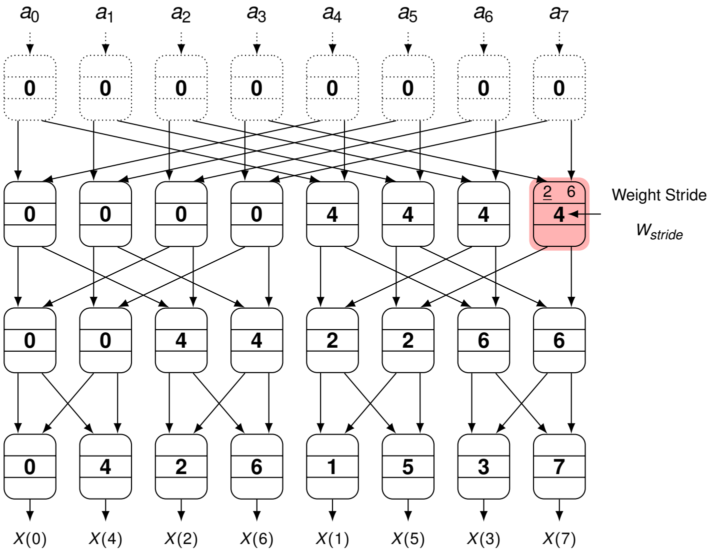

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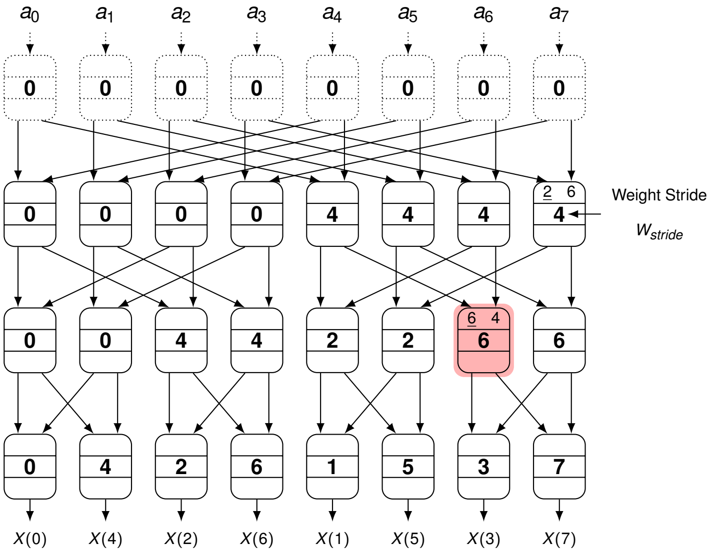

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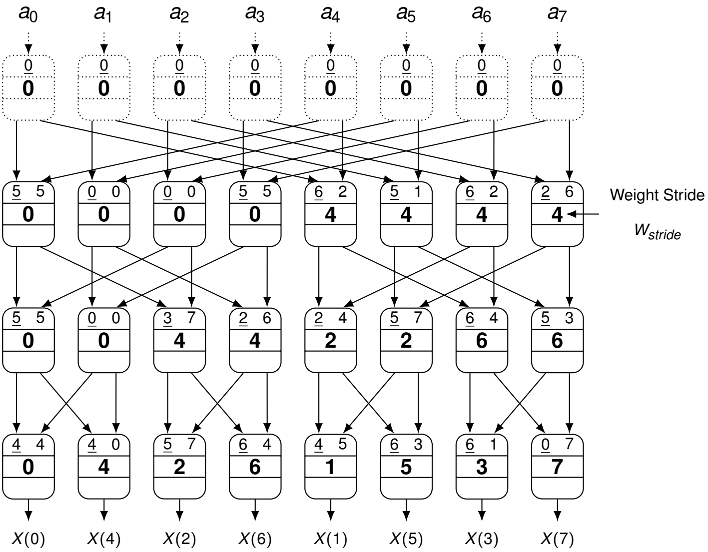

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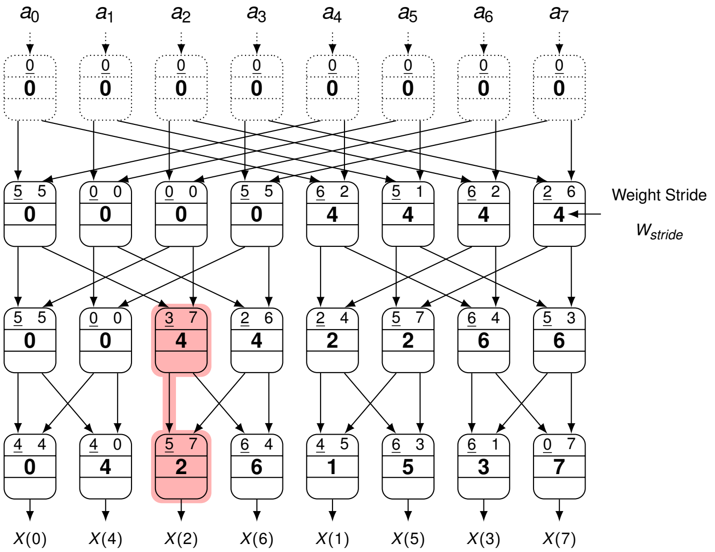

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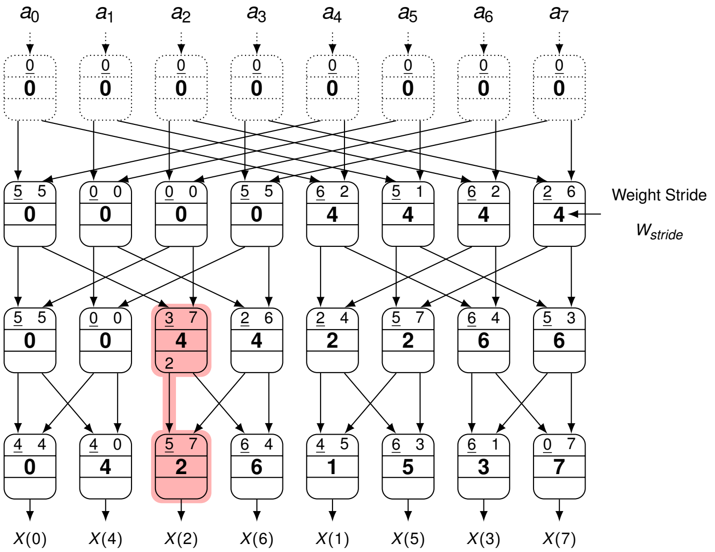

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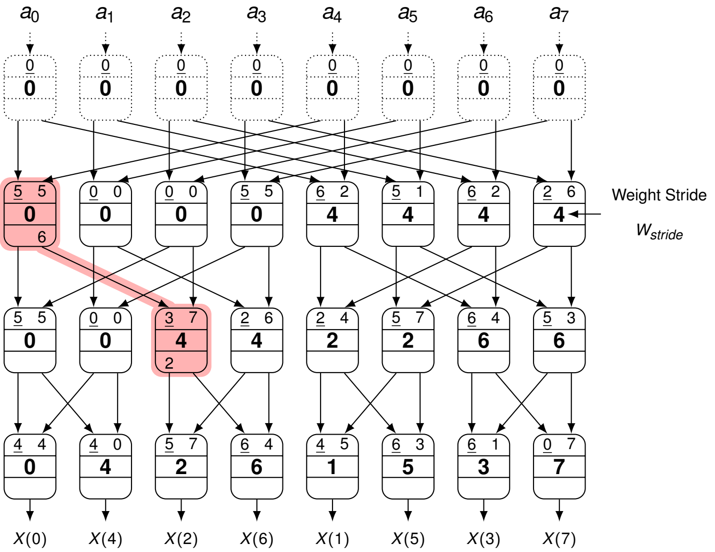

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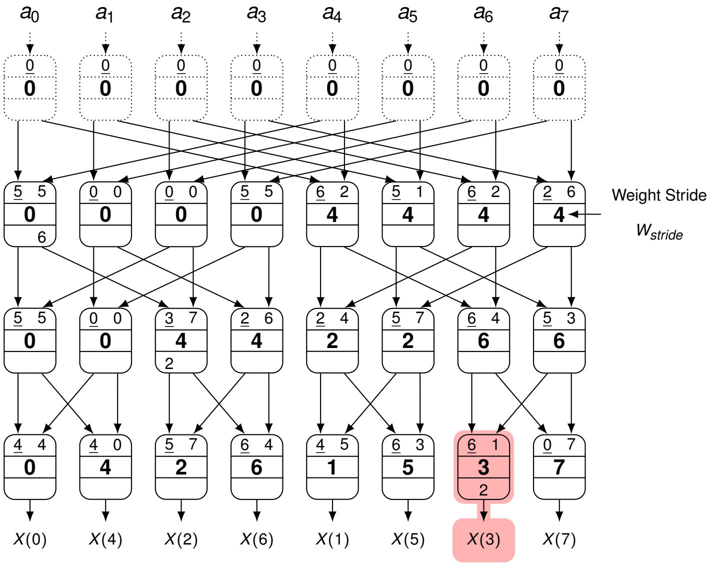

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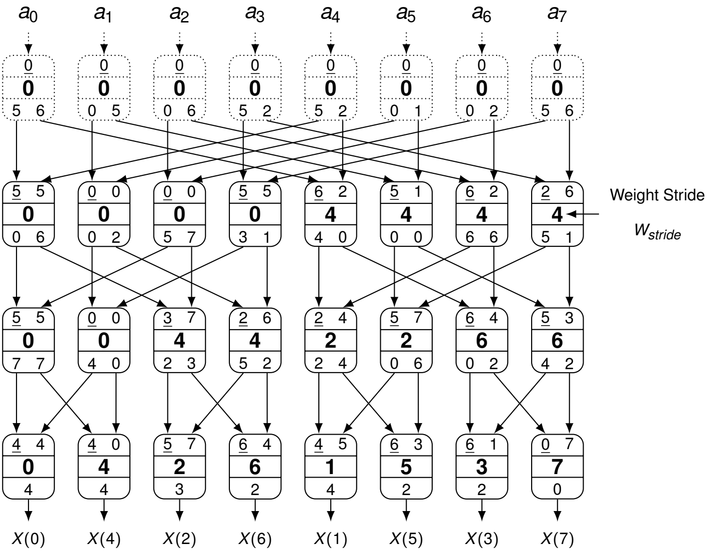

How to Generate a Random Member FFT Algorithm

Input: Size-n flowgraph with labeled invariants
Output: Size-n flowgraph with twiddle factors assigned
foreach $nd \in flowgraph$ **do**
  if $nd.stride \neq n$ **then**
    $nd.W_{base} \leftarrow rand() \pmod{n}$
    $nd.rW_{base} \leftarrow nd.W_{base} + nd.W_{stride} \pmod{n}$
  else
    $nd.W_{base} \leftarrow 0$
foreach $nd \in flowgraph$ **do**
  if $nd.stride \neq n$ **then**
    $nd.lp.tfp \leftarrow nd.W_{base} - nd.lp.W_{base} \pmod{n}$
    $nd.rp.tfp \leftarrow nd.rW_{base} - nd.rp.W_{base} \pmod{n}$
  if $nd.stride = 1$ **then**
    $nd.tfp \leftarrow 0 - nd.W_{base} \pmod{n}$

Outline

Searching a Family of FFT Algorithms

- All family members are not equally desirable
  - Some require fewer FLOPs
  - Others may require less communication of twiddle factors
  - Need a way to search and find desirable members
- How many family members are there?
  - $2^{n \log_2 n \log_2 n}$
  - For a 256-point FFT: 2^{16384}
  - Only 1 in 2^{18432} chance of guessing correct twiddle factors
  - Estimated atoms in the universe is 2^{264}
  - Fastest supercomputer performs 2^{144} FLOPS

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A First SMT Formulation

- Directly cast "Random Member Algorithm" as SMT
- Must also calculate FLOP count directly in SMT model
  - Psuedo-Boolean constraint
  - Nearly balanced add tree in implementation
  - ITE tree did not work well
  - Did not implement sorter-based techniques
- Size-32 455 FLOP search UNSAT in 30 seconds
- Time-out of 24 hours reached for size-64 1159 FLOP search

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Sample SMT 1.2 Code

```
(benchmark example1
:logic QF_BV
...
:extrafuns ((Wb_2_1_6 BitVec[4]))
:extrafuns ((Wb_2_1_14 BitVec[4]))
...
:formula
...
(let (?Wb_16_14_0 bv0[4])
...
(let (?rWb_2_1_6 (bvadd Wb_2_1_6 bv6[4]))
...
(let (?ltfp_4_1_12 (bvsub Wb_2_1_6 ?Wb_4_1_12))
(let (?ltfp_4_3_12 (bvsub ?rWb_2_1_6 ?Wb_4_3_12))
...
(flet ($c0_4_1_12 (= (extract[1:0] ?ltfp_4_1_12) bv0[2]))
(flet ($c4_4_1_12 (and (= (extract[0:0] ?ltfp_4_1_12) bv0[1]) (not $c0_4_1_12)))
(flet ($c6_4_1_12 (not (= (extract[0:0] ?ltfp_4_1_12) bv0[1])))
(let (?cost_4_1_12 (ite $c6_4_1_12 bv6[4] (ite $c4_4_1_12 bv4[4] bv0[4])))
...
(let (?totalcost (bvadd ?cost_2_2_1 (bvadd ?cost_4_1_12 ?cost_4_3_12)) ...
(flet ($maxcost (bvule ?totalcost bv22[4]))
$maxcost
)...)
```

What is Required for Larger Problems?

- A size-32 naïve formulation can be solved easily
  - Interesting results happen at size-256 and larger
- To solve larger problems:
  - Exclude cost symmetries
  - Share twiddle factors
  - Partition
  - Exclude local symmetries
- Will only present partitioning

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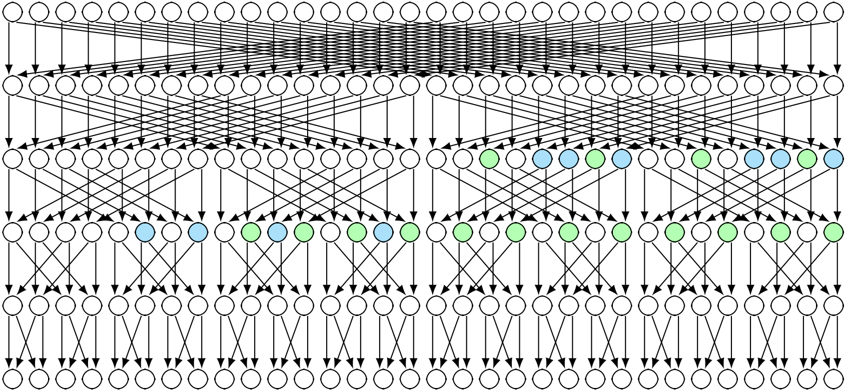

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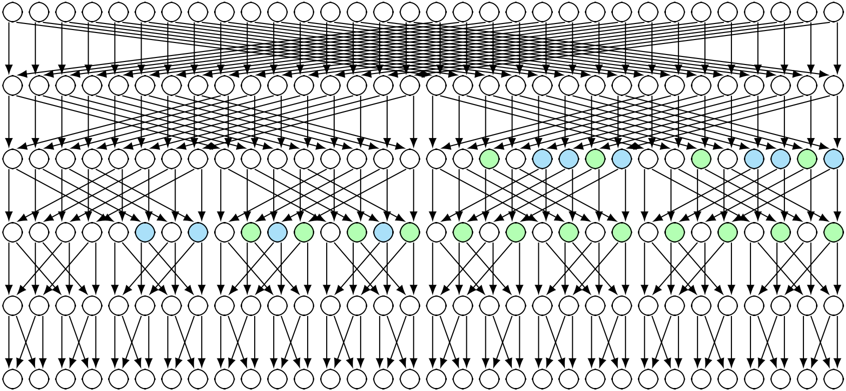
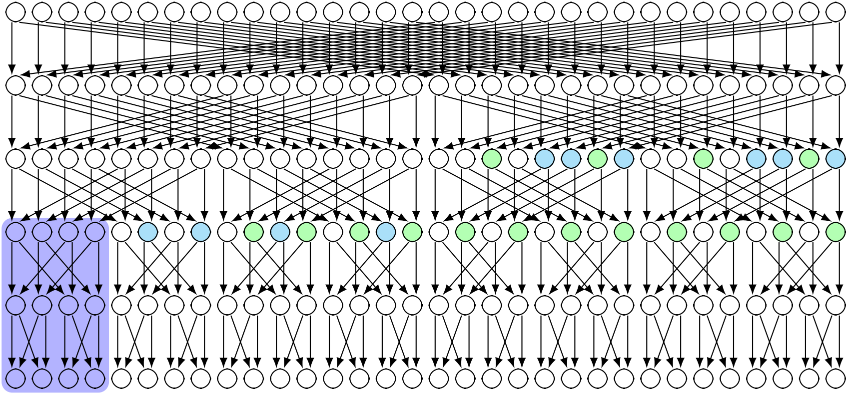


Terminal Weights Known

Partitioning



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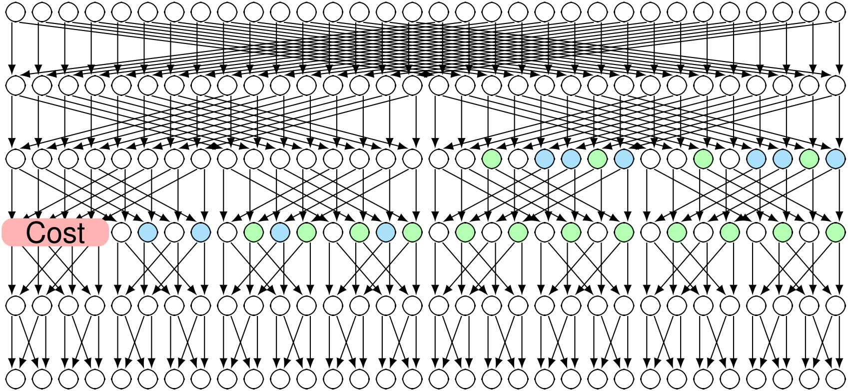

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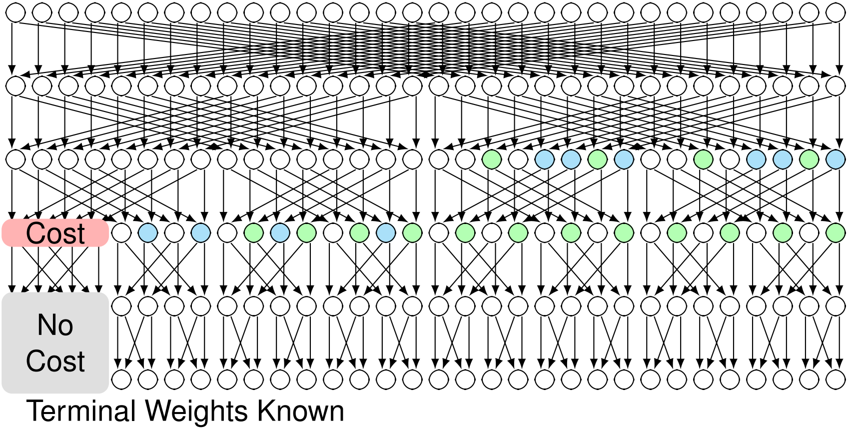

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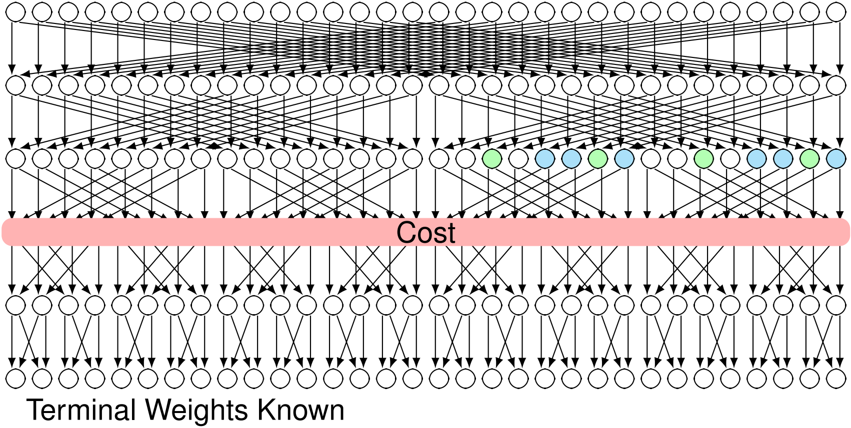

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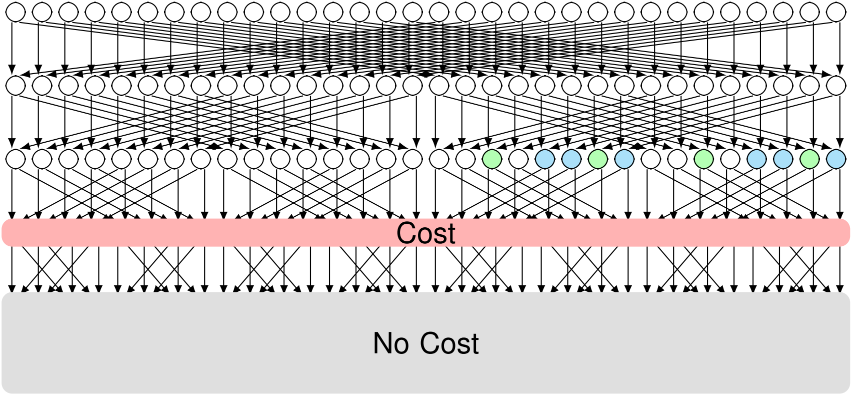

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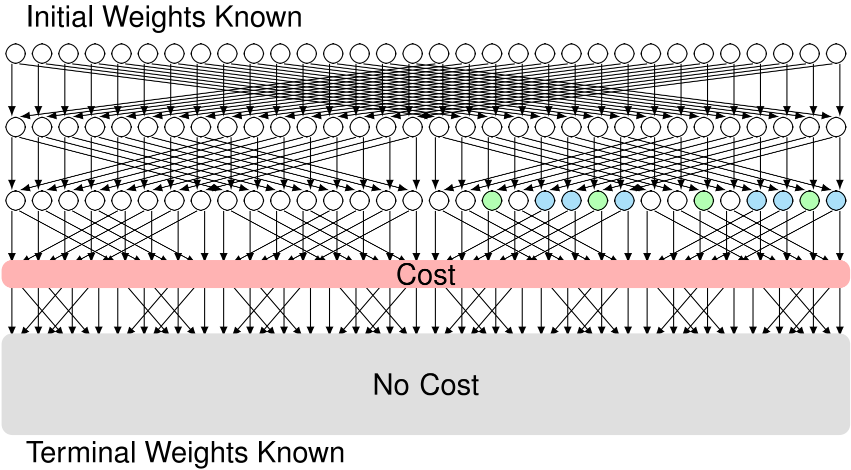

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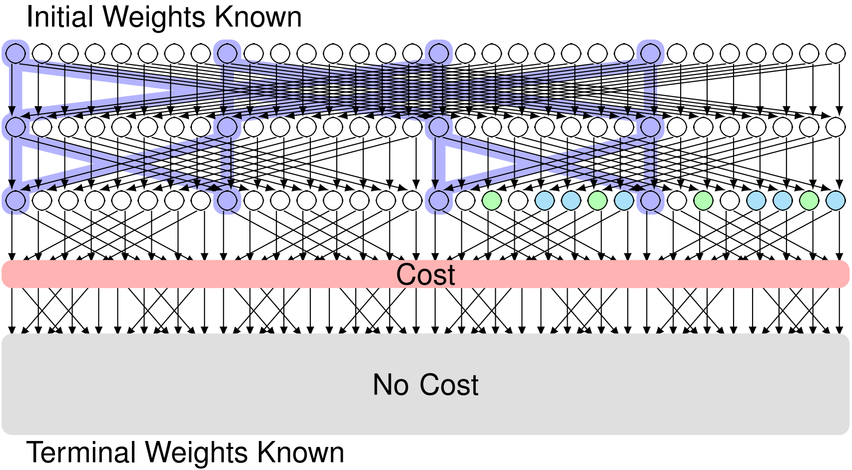

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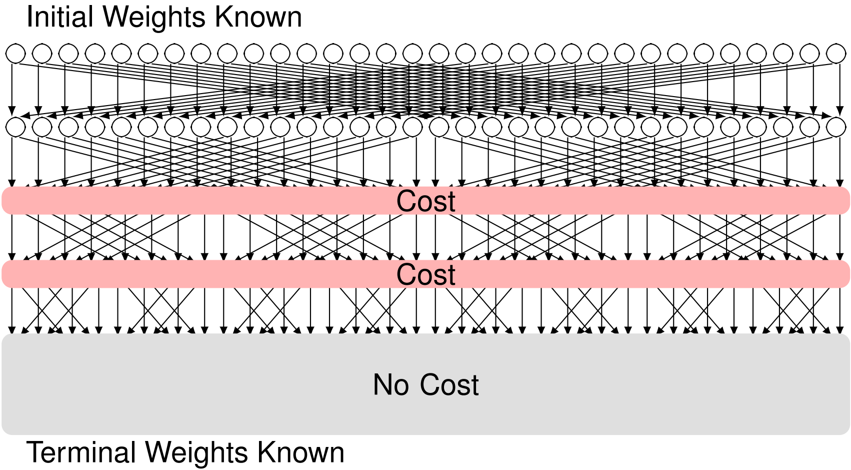

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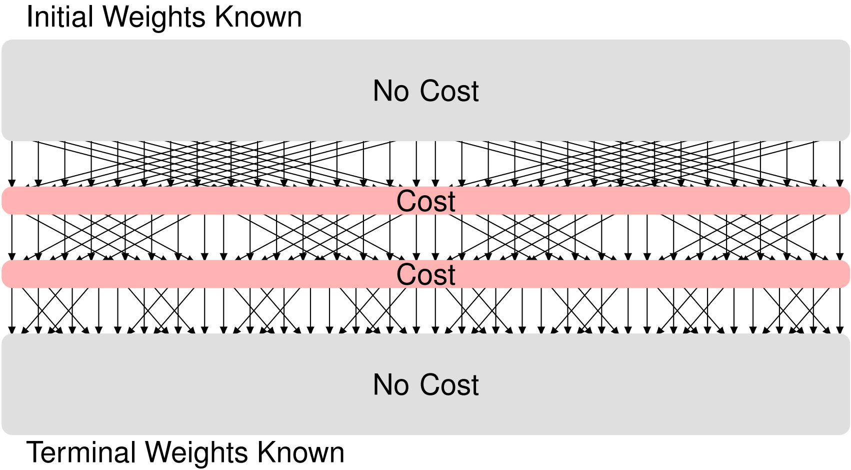

Partitioning

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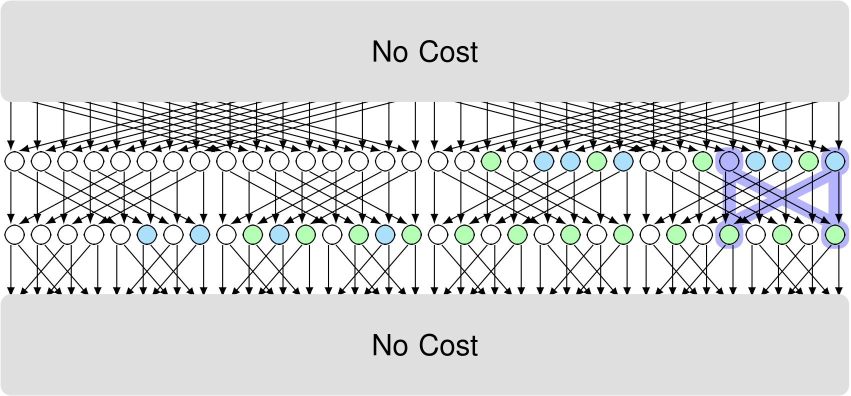

Terminal Weights Known

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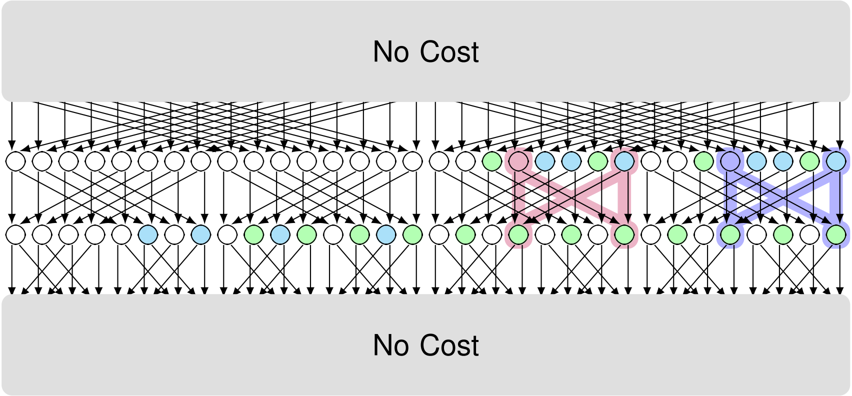

Terminal Weights Known

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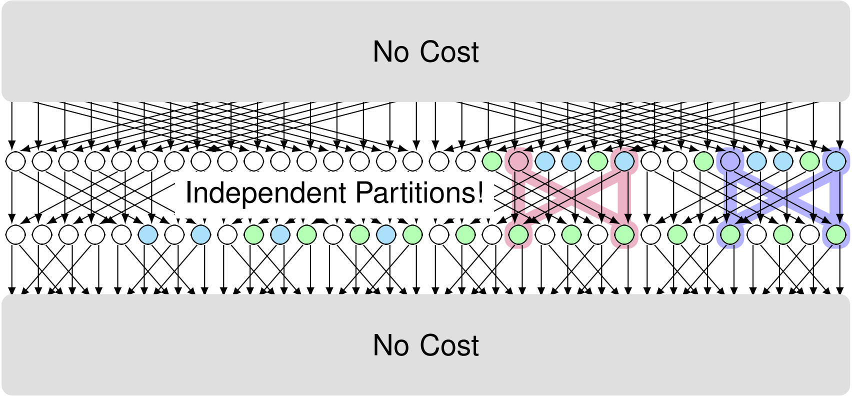

Terminal Weights Known

Partitioning

- Hardest partitions for size-256 are 8 size-16
- Solution space for size-16 flowgraph is 2^{192}
- Size-16 best FLOP count minus one UNSAT in 5 seconds
- Size-256 6616 FLOP search SAT for all partitions in 8 seconds
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Outline

1. The Fast Fourier Transform
2. Generating a Family of FFT Algorithms
3. Searching a Family of FFT Algorithms
4. Results and Conclusions

Results

	Tangent $\lvert\omega_n^*\rvert = *$	Split-Radix $\lvert\omega_n^*\rvert = 1$	SMT Search $\lvert\omega_n^*\rvert = 1$			
			Satisfiable		Unsatisfiable	
FFT Size	**FLOPs**	**FLOPs**	**FLOPs**	**time(s)**	**FLOPs**	**time(s)**
32	456	456	456	1.4×10^{-1}	455	1.5×10^{-1}
64	1152	1160	1160	3.1×10^{-1}	1159	3.3×10^{-1}
128	2792	2824	2824	9.3×10^{-1}	2823	1.1×10^{0}
256	6552	6664	6616	8.3×10^{0}	6615	5.0×10^{1}
512	15048	15368	15128	3.9×10^{4}	15127?	$>1 \times 10^{6}$

- Found FFT algorithms with restricted set of twiddle factors
- Boolector was our SMT solver of choice
- Found no computation advantage to using SMT over SAT
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- SMT proof of the lowest operation count for classes of Fourier transforms
  - Flowgraph structure of common FFTs
  - All complex multiplication by n^{th} roots of unity
- Found new FFTs with lower FLOP count than split-radix
  - Undiscovered in past 40 years despite intense study
- Next Steps
  - Impose regularity to develop traditional algorithm
  - Develop other search objectives
  - Relax graph structure
  - Relax constraint that twiddle factors must have modulus one
  - Apply to matrix multiplication
- Full paper to appear in JSAT
  - Preprint at www.arXiv.org
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